The Gamow Vectors and the Schwinger Effect

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We introduce a 'proper time' formalism to study the instability of the vacuum in a uniform external electric field due to particle production. This formalism allows us to reduce a quantum field-theoretic problem to a quantum mechanical one in a higher dimension. The instability results from the inverted oscillator structure which appears in the Hamiltonian. We show that the 'proper time' unitary evolution splits into two semigroups. The semigroup associated with decaying Gamov vectors is related to the Feynman boundary conditions for the Green functions and the semigroup associated with growing Gamov vectors is related to the Dyson boundary conditions.

1. INTRODUCTION

The history of the unification of relativity and quantum mechanics began with the formulation of one-particle wave equations for irreducible representations of the Lorentz group (Klein–Gordon, Dirac, etc.). However, this attempt was shown to be unsuccessful, due mainly to the appearance of negative-energy solutions, which was originally considered as a strong anomaly. Quantum field theory (QFT) arose as the solution of such difficulties shedding light on many new results, for example, particle creation produced by classical fields. In field-theoretic language this effect originates from the instability of the vacuum of the matter field, which privileges pair creation with respect to pair annihilation, due to the nonsymmetrical choice of vacuum, compatible with the idea of the Dirac sea (all negative-energy levels are filled). Such a process, which posed a problem to the one-particle interpretation of the theory can nevertheless be pictured as the possibility of tunneling from

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negative- to positive-energy states under the influence of the potential barrier produced by the external field (Lifshitz and Pitaevski, 1971; Brout et al., 1991; Brout and Parentani, 1992). This picture is extremely helpful for clarifying physical ideas, so we begin by explaining it in Section 2.

The dissipative behavior of tunneling processes in nonrelativistic quantum mechanics has been extensively studied, and with this instability can be associated Gamow vectors, which serves as a representation of the exponential decay law (Bohm et al., 1989). The aim of this work is to study the intrinsic instability of the particle production mechanism. However, it is not clear how to introduce Gamow vectors in the standard first-quantized theory nor at the level of second quantization. So we consider an alternative point of view given by consistent a first-quantized formulation of relativistic quantum mechanics (RQM) based on a proper time method.[2] The key idea of this formulation is to replace the Dirac solution to the negative energy problem by the Stueckelberg interpretation (Stueckelberg, 1941, 1942; Feynman, 1948, 1949a, b), i.e., to consider (negative-energy) particles traveling backward in time as antiparticles, introducing a fifth invariant parameter for labeling the evolution of the system. This formalism has the advantage that one can deal with a 'one-particle' configuration space of higher dimension instead of the infinite-dimensional problem of QFT. We discuss the main outlines of this formalism for charged scalar particles in Section 3; then we consider the problem of these particles in an external constant and homogeneous electric field (Section 4) and then introduce rigged Hilbert spaces associated with the choice of the boundary conditions for the propagators (Feynman and Dyson) and the corresponding proper time evolution of Gamow vectors generated by a semigroup of unitary operators (Section 5), closely related to a subjacent upside-down oscillator structure.

2. THE PHYSICAL PICTURE

Since the works of Heisenberg, Euler, and Kockel (Euler and Kockel, 1935; Heisenberg and Euler, 1936), we know that the vacuum is not an inert object. In fact it behaves as a dielectric in the presence of an external electric field, and such an effect introduces nonlinear corrections to the Maxwell equations, to be discussed in Section 4. Vacuum polarization corresponds to a virtual pair creation and annihilation process; however actual pair creation can occur in the presence of external fields. Sauter was the first in estimate

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[2] From the classical work of Fock (1937), Nambu (1950), Feynman (1950, 1951), and Schwinger (1951) the proper time formalism was used for computing the effective action and studying the problem of particle creation in external fields. In connection with this work see, for example, Hartle and Hawking (1976), Rumpf (1977), Rumpf and Urbantke (1978), Stephens (1988), Brout et al. (1991), and Brout and Parentani (1992).

the probability of creating a pair in presence of a constant homogeneous electric field (Lifshitz and Pitaevskii, 1971). His reasoning was at the level of the semiclassical limit of a first-quantized theory. However, it is so clear that the simple translation of his physical picture to the intricate theoretical framework of quantum field theory shed light on the vacuum instability problem. Before discussing the Sauter derivation, let us recall the dictionary which allows us to translate the concepts of first quantization to QFT.

Often, textbooks repeat that we can do relativistic quantum mechanics of the Klein–Gordon field consistently and we can go straightforwardly to quantum field theory. Certainly, this is not true. In this position we see a theoretical prejudice against negative energies and indefinite metric spaces. The first obstacle was cleared up by Stueckelberg (1941, 1942) and Feynman (1948, 1949a, b) and a comprehensive review of the second can be found in the work of Feshbach and Villars (1958). Now let us concentrate in the Stueckelberg and Feynman ideas using the classical picture derived from the Klein–Gordon equation ($\hbar = c = 1$)

$$(D^\mu D_\mu + m^2)\psi = 0 \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$ is the gauge-covariant derivative and $\psi(x)$ is the complex scalar field. Using the WKB approximation in (1)

$$\psi(x)_{\text{WKB}} = ae^{-iS(x)} \tag{2}$$

we obtain the Hamilton–Jacobi equation in the classical limit

$$(\partial_\mu S - eA_\mu)(\partial^\mu S - eA^\mu) = m^2 \tag{3}$$

where S is identified with the classical action and its gradient with the canonical momentum

$$p_\mu = \partial_\mu S \tag{4}$$

Equation (3) is the mass-shell condition for spinless particles and it is equivalent to the proper time velocity constraint ($\eta_{\mu\nu} = \text{diag}\{+1, -1, -1, -1\}$)

$$\eta_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 1 \tag{5}$$

due to the proportionality between the velocity and the kinematical momentum $\pi_\mu = p_\mu - eA_\mu$

$$m \frac{dx^\mu}{ds} = \pi_\mu \tag{6}$$

The essence of the Stueckelberg–Feynman interpretation for antiparticles rests on a fundamental symmetry which appears in the equations of motion

of charged particles when we reexpress them as functions of the proper time. Then we can see from equation (3) and (6) that the proper time reversal of the motion (which inverts the four-velocity, the action, and the canonical momenta) is equivalent to a charge conjugation in these equations. Then classical motions in which particles go backward in coordinate time $(dx^0/ds < 0)$, negative-kinetic-energy states according to equation (6), can be reinterpreted as charge-conjugated particles (antiparticles) going forward in time. It is important to remark that our argument can be extrapolated to the semiclassical level. In fact, we see from the WKB wave function that the operation which conjugates the charge in the Klein–Gordon equation (the complex conjugation of the wave function) is equivalent to the operation which reverses the motion in the proper time (the inversion of the action). However, a pure quantum explanation of this fundamental symmetry requires the PFT we develop in Section 3, in which the concept of proper time is introduced at the quantum level. At the moment we only use the fact that the sign of the kinetic energy is the gauge-invariant quantity which classifies particle and antiparticle states.

Standard QFT is nothing else than the second quantization (many-particle theory) after replacing the level of the natural vacuum (all positive-kinetic-energy states unfilled) by the notion of a vacuum which rest on Dirac sea of fermions (all negative-kinetic-energy states filled). Therefore pair creation in field-theoretic language (a positive-kinetic-energy state plus a hole in the sea) corresponds to a transition from a negative-kinetic-energy state to a positive one in the first-quantized theory. Note that the charge is also negative in a negative-kinetic-energy state, in such a way that it leaves a positive charged hole, in field-theoretic language. It becomes positive when we reverse the arrow of time according to the Stueckelberg–Feynman interpretation.

With this ideas in mind, we follow the Sauter semiclassical argument. Suppose that we have a homogenous constant electric field $\mathbf{E} = E e_3$. Then the classical expression for a relativistic charged particle of total energy p_0 is

$$p_0 = -\sqrt{p^2 + m^2} + eEx^3 \qquad (7)$$

where we have assumed that the kinetic energy is initially negative and the charge is $-e$ $(e > 0)$. Suppose that we have such an incoming particle (the outgoing antiparticle in field-theoretic language) in the positive direction of the coordinate x^3 $(\mathbf{p} = p e_3)$. We see that it decelerates in the presence of the electric field in such a way that the kinetic term in (7) decreases as the potential energy increases. There is a turning point $a = (p_0 - m)/eE$ at the coordinate x^3 for this classical particle. But, quantum mechanically, the momentum p can become imaginary $(p = ik)$, allowing the tunneling of this particle through the barrier. Inside the barrier we have the dispersion relation

$$k = \sqrt{m^2 - (p_0 - eEx^3)^2} \qquad (8)$$

for the semiclassical wave associated with the particle. On the other side of the barrier, we assume that the dispersion relation is

$$p_0 = +\sqrt{p^2 + m^2} + eEx^3 \qquad (9)$$

for the outgoing wave. It corresponds to a classical particle of positive kinetic energy, which begins its movement at the point $b = (p_0 - m)/eE$.

We can estimate the probability of crossing the barrier using the Gamow formula derived from the WKB approximation

$$w = A \exp(-2 \int_a^b k(x^3) \, dx^3) \qquad (10)$$

where A is an undefined proportionality constant. Evaluating the integral $[t = (p_0 - eEx^3)/m]$

$$\int_a^b k(x^3) \, dx^3 = -\frac{m^2}{eE} \int_{+1}^{-1} \sqrt{1 - t^2} \, dt = \frac{\pi m^2}{2eE}$$

we finally have

$$w = A \exp\left(-\frac{\pi m^2}{eE}\right) \qquad (11)$$

In Section 4 we will see that the exact Schwinger calculation can be fitted by a Sauter expression upon choosing $A = e^2E^2/8\pi^3$. But the merit of this heuristic derivation lies in its physical picture. The vacuum of field theory (the Dirac sea) in presence of an electric external field is like an infinite unstable nucleus which disintegrates by nucleon emission (our first-quantized negative-energy states going through the barrier). Another interesting picture can be traced at the classical level by solving the proper time equations of motion (Garcia Alvarez *et al.*, 1997) (Lorentz force law)

$$\frac{d^2x^\mu}{ds^2} = \frac{e}{m} F^{\mu\nu} \frac{dx_\nu}{ds}$$

For the case of constant electric field discussed above, adequately choosing the initial conditions, we obtain the hyperbolic trajectories

$$\begin{cases} x = \dfrac{m}{eE} \cosh\left(\dfrac{eEs}{m}\right) \\[2mm] t = \dfrac{m}{eE} \sinh\left(\dfrac{eEs}{m}\right) \end{cases}$$

Usually, in textbooks on electrodynamics, one finds only one of the branches of the hyperbola, depending on the sign of the charge (e.g., Bredov

et al., 1984). In this picture both particles and antiparticles evolve in the ordinary direction of coordinate time.[3] However, for a given charge the two solutions follow from the equations of motion in proper time (which is analogous to what happens with the relativistic wave equations, which admit both positive- and negative-energy solutions). Moreover, according to the Stueckelberg and Feynman interpretation, one branch corresponds to the particle and the other to the antiparticle. Now, let us compare the natural length scale which appears in the classical problem (the minimal separation between the two branches of the hyperbola, $2m/eE$) with the quantum scale of the Compton wavelength $(1/m)$ given by the uncertainty principle. We see that when these scales have the same order, particle and antiparticle trajectories overlap, increasing the probability of creating a pair. In this case the strength of the electric field must be of the order $E \sim e/2m^2$ according to the Sauter estimation for the tunneling process.

   Closing this section, we note that our initial vacuum is unstable under the pair creation process and not under pair annihilation. This is because we have filled all negative-energy states. In Section 3 we will see that this is related to the analytical structure of the Feynman propagator.

3. THE PROPER TIME FORMALISM FOR THE CHARGE SCALAR FIELD

   In Section 2 we saw that proper time can be introduced in the standard formulation only at the semiclassical level. The lack of such an invariant evolution parameter is the main difficulty in reconciling the antagonistic formalisms of quantum mechanics and the theory of relativity (Gaioli and Garcia Alvarez, 1994). In fact, while relativity deals with the space-time coordinates on an equal footing, quantum mechanics privileges an external absolute parameter to label the evolution of the state of the system. Therefore, in the second case, "time" should have the properties of a c-number, unlike in the first case, where, since the spatial coordinates are raised to the status of operators, Lorentz transformations should impose this character on the temporal coordinate as well. Thus, this dual role of "time" generates trouble in RQM. This is basically why a one-particle theory in the usual formulation of RQM runs into several conceptual difficulties.

   In the 1940s a PTF was developed for a possible unification between special relativity and quantum mechanics within the framework of a consistent one-particle theory.[4] However, the price paid for the PFT was giving up the

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[3] This corresponds to the standard interpretation of QFT, which in the canonical formulation uses the coordinate time as evolution parameter.

[4] See Fanchi (1993) and Aparicio *et al.* (1995a, b) for a review of different proposals.

concept of a definite mass state. In fact, suppose that we want to develop a relativistic classical theory in an explicit covariant way, putting the spatial coordinates and time on the same footing. We immediately notice that the Poisson bracket

$$\{x^\mu, p^\nu\} = -\eta^{\mu\nu} \tag{12}$$

is incompatible with the classical mass constraint

$$p^\mu p_\mu = m^2 \tag{13}$$

So the classical mass constraint (13) must be removed from the PTF, although nevertheless the standard RQM is recovered on-shell. Notice also that the Poincaré algebra must be enlarged to contain relation (12). Among the different extensions of the Poincaré group, whose algebra includes the canonical commutation relation corresponding to (12), the five-dimensional Galilei group (Aghassi et al., 1970)

$$x^\mu \mapsto x^\mu + v^\mu \tau$$

$$\tau \mapsto \alpha\tau + \beta$$

$$x^\mu = L^\mu_\nu x^\nu + a^\mu \quad \text{(Poincaré)}$$

is the alternative most frequently used in the literature.[5] We consider this formulation in this work.

Actually the parameter τ, which gives the name to the formalism, is *a priori* independent of the classical proper time, but it can be related to it in the classical limit on the mass shell. In the case of the Galilean version of PTF, this parameter is nothing else than the parameter of the generalized Galilean boost, and it plays the role of a Newtonian time (Horwitz and Piron, 1973; Sonego, 1991). The temporal coordinate has a different status, and it is promoted to the rank of operator in accordance with the classical Poisson bracket (12). This dissociation of roles solves the conflict exposed above. Of course, the notions of simultaneity and causality are correspond to Newtonian time, but standard relativistic notions in coordinate time are reobtained on shell [e.g., we will see that the retarded (causal) propagator of the PFT on-shell is reduced to the Feynman one].

In this way the formalism closely copies the general outlines of nonrelativistic quantum mechanics, furnishing the theory with a well-known structure. Let us outline it for the case of the spinless relativistic point particle.

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[5]The de Sitter alternative is discussed in Garcia Alvarez and Gaioli (1997).

Let $\{|x^\mu\rangle\}$ ($\mu = 0, 1, 2, 3$) be the basis of localized states of the position operator x^μ for the charge of the system. This basis spans a linear space endowed with a scalar product,

$$\langle \Phi | \Psi \rangle = \int d^4x \; \Phi^*(x^\mu)\Psi(x^\mu) \tag{14}$$

satisfying the normalization and completeness conditions

$$\langle x^\mu | y^\mu \rangle = \delta(x^\mu - y^\mu), \qquad \int d^4 \; x|x^\mu\rangle\langle x^\mu| = I$$

In this coordinate representation the state of the system is represented by a wave function belonging to a four-dimensional Hilbert space, defined on the space-time manifold. The position operator x^μ and its canonical conjugate variable, the momentum p_μ, satisfy

$$[x^\mu, p^\nu] = -i\eta^{\mu\nu} \tag{15}$$

and are given by

$$\langle x|p_\mu|\Psi \rangle = i\partial_\mu \Psi(x)$$

$$\langle x|x^\mu|\Psi \rangle = x^\mu \Psi(x)$$

[In equation (15), $-i$ was chosen to preserve the sign in the ordinary relations for the spatial part.]

In the Schrödinger picture,

$$|\Psi(x^\mu, \tau)|^2$$

represents the probability density for the system to be at the space-time point x^μ at "instant" τ. The wave function evolves with a Schrödinger equation,

$$-i\frac{d}{d\tau} |\Psi(\tau)\rangle = H|\Psi(\tau)\rangle \tag{16}$$

where

$$H = \frac{\eta^{\mu\nu}\pi_\mu\pi_\nu}{2M}$$

is a super-Hamiltonian and M a super-mass parameter.

The time-reversal operator in the Wigner sense coincides with the charge conjugation operator

$$C\Psi(x, \tau) = \Psi^*(x, -\tau) \tag{17}$$

This fact naturally introduces the Stueckelberg–Feynman interpretation into the formalism. Notice that from the Heisenberg equation of motion we have

$$\frac{dx^\mu}{d\tau} = \frac{\pi^\mu}{M}$$

so, as mentioned above, at the classical level on the mass shell the parameter τ is proportional to the classical proper time

$$d\tau^2 = \left(\frac{M}{m}\right)^2 ds^2$$

From now on, we rescale the time, writing $s = (1/2M)\tau$, using the Schwinger notation, which is more familiar in the literature. s must not be confused with the classical proper time. The Schrödinger equation now reads

$$i\,\frac{\partial \psi(x, s)}{\partial s} = D^\mu D_\mu \psi(x, s) \tag{18}$$

In this case the super-Hamiltonian is rescaled to

$$H = \eta^{\mu\nu}\pi_\mu\pi_\nu = \pi^2$$

The super-Hamiltonian as well as position and momentum operators are Hermitian in the inner product (14).

A stationary solution of (16) is

$$\Psi(x^\mu, s) = e^{im^2 s}\psi_{m^2}(x^\mu)$$

where $\psi_{m^2}(x^\mu)$ is a solution of the generalized Klein–Gordon equation

$$D^2\psi_{m^2}(x^\mu) + m^2\psi_{m^2}(x^\mu) = 0$$

which is reinterpreted as an *eigenvalue* equation (the mass eigenvalue m^2 is real, but not restricted *a priori* to be positive).

We can see that the theory developed is formally identical to ordinary quantum mechanics. Then, all of what we have learned from this theory can be rewritten in the PTF. For example, let us consider the resolvent of the free Hamiltonian operator

$$R(z) = \frac{1}{H - z} = \frac{1}{p^2 - z}$$

We see that $R(z)$ is analytic in all the complex plane except for a cut along the positive real axis. $R(z)$ is the extension to the complex plane of the inhomogeneous Green function of the Klein–Gordon equation

$$(p^2 - m^2)G(m^2) = 1$$

The limiting values of $R(z)$ approaching the positive real axis define two analytic functions in the lower (upper) half-plane. These functions are those defined in QFT by adding a negative (positive) small imaginary part to m^2 in order to give sense to the formal expression

$$G_{\pm}(m^2) = \frac{1}{p^2 - m^2 \pm i\epsilon}, \qquad \epsilon > 0$$

In our notation, plus and minus correspond to the Feynman and Dyson propagators, respectively,

$$G_{F(D)}(x, y) = \langle x|G_{\pm}(m^2)|y\rangle$$

These functions can be analytically continued across the cut to the second Riemann sheet on the upper (lower) complex half-plane, defining two analytic functions for all $z \in C$.

As is well known, the Feynman (Dyson) propagator can be obtained in the first-quantized theory as the inhomogeneous Klein–Gordon Green function which propagates positive (negative)- and negative (positive)-energy states forward (backward) and backward (forward) in time, respectively. In the language of QFT, the Feynman propagator can be obtained as the mean value in the vacuum state of the time-ordered product of field operators

$$G_F(x, y) = i\langle 0|\theta(x^0 - y^0)\psi_{m^2}(x)\psi_{m^2}^{\dagger}(y) + \theta(y^0 - x^0)\psi_{m^2}^{\dagger}(y)\psi_{m^2}(x)|0\rangle$$

(we can write an analogous expression for the Dyson propagator, defining a vacuum in which all positive-energy states are filled).

As we will see, Feynman and Dyson boundary conditions also have an interpretation in the off-shell theory. The two limiting values of $R(z)$ are connected with the retarded and advanced Green functions of the Schrödinger equation. In fact, applying the formal identity

$$\frac{1}{a \pm i\epsilon} = \mp i \int_{-\infty}^{\infty} \theta(\pm s)e^{i(a \pm i\epsilon)s} \, ds$$

for $a = H - m^2$, we see that

$$G_{\pm}(m^2) = \frac{1}{H - m^2 \pm i\epsilon} = \pm i \int_{-\infty}^{\infty} \theta(\pm s)e^{is(H - m^2 \pm i\epsilon)} \, ds \qquad (19)$$

Equation (19) in coordinate representation reads

$$G_{F(D)}(x, y) = \int_{-\infty}^{\infty} G_{\pm}[x(s), y(0)]e^{-s(m^2 \pm i\epsilon)} \, ds \qquad (20)$$

with

$$G_{\pm}[x(s), y(0)] = \mp i \; \theta(\pm s)\langle x|e^{isH}|y\rangle$$

representing the retarded and advanced solutions of the Schrödinger equation
of the off-shell theory. Equation (20) is the analogue of the relation between
time-dependent and time-independent Green functions in nonrelativistic quan-
tum mechanics. The Fourier integral in s selects a particular value of m^2 of
the indefinite mass theory, and tell us that the Feynman (Dyson) propagator is
the time-independent Green function corresponding to the retarded (advanced)
one in the off-shell theory.

Summarizing, we have seen that (a) the analytical continuation of the
resolvent to the upper complex half-plane in the second sheet, (b) the boundary
conditions of the on-shell Green function according to the Stueckelberg
interpretation, (c) the choice of the vacuum according to the Dirac sea idea,
and (d) the causal (retarded) boundary conditions of the Green function of
the off-shell theory are different aspects of the same thing. In the next sections
we show that in the case of the external field problem one of the two possible
analytical continuations of the resolvent (the one associated with the Feynman
boundary conditions) corresponds to decaying Gamow vectors in proper time,
which are related to the instability of the vacuum of field theory under
pair creation.

4. THE HEISENBERG–EULER EFFECTIVE ACTION AND PARTICLE CREATION

The effective action S_{eff} due to the interaction of the vacuum current
with the external field such that

$$\delta S_{\text{eff}} = \int dx^4 \; \langle 0|J^{\mu}|0\rangle \delta A_{\mu}$$

leads to the Heisenberg–Euler corrections of the Maxwell equations. As
shown by Schwinger (1951), it can be obtained in the proper time formalism
by computing

$$S_{\text{eff}} = \int L_{\text{eff}}(x) \; dx^4$$

where

$$L_{\text{eff}}(x) = -i \int_0^{\infty} \frac{ds}{s} \langle x|e^{iHs}|x\rangle e^{-i(m^2 - i\epsilon)s} + C(x) \qquad (21)$$

is the effective Lagrangian and $C(x)$ is an additive constant determined in
such a way that $L_{\text{eff}}(x) = 0$ in the absence of external fields. We see from

equation (21) that we have again reduced a field-theoretic problem (the calculation of the effective action) to a quantum mechanical one in a higher dimension. In fact, now our problem consists in evaluating the matrix element $\langle x|e^{iHs}|x\rangle$. Moreover, we can reinterpret it as the persistence amplitude of the off-shell particle to remain at the point x^μ of the space-time, and $L_{\text{eff}}(x)$ as the on-shell correlate of this amplitude per unit 'proper time.'

The matrix element $\langle x|e^{iHs}|x\rangle$ can be evaluated by path integrals or, in the Heisenberg picture of the canonical formulation, by means of an ingenious procedure developed by Schwinger departing from the integration of the Heisenberg equations of motion. Here we solve the eingenvalue problem in the Schrödinger picture. The reason is that through this procedure it is clearer that the proper time evolution splits into two semigroups.

Let us consider again the problem of a homogeneous and constant electric field. Using the conventions of Section 3, the super-Hamiltonian reads

$$H = \pi^2 = (p_0 + eEx^3)^2 - p_3^2 - p_1^2 - p_2^2 \tag{22}$$

We see that it can be split into the Hamiltonian of an upside-down harmonic oscillator plus the Hamiltonian of two free particles in the coordinates of the plane perpendicular to the electric field

$$H = -H_{\text{osc}} - p_1^2 - p_2^2$$

$$H_{\text{osc}} = p_3 + (ieE)^2(x^3 + p_0/eE) \tag{23}$$

The eingenvalue problem for the free-particle part has the standard plane-wave solution $\langle x^1 x^2|p_1 p_2\rangle$. The inverted harmonic oscillator has pure imaginary frequency $\omega/2 = \pm ieE$. In Section 5 we see in detail that the two signs correspond to the splitting of the proper time evolution of this unstable system toward the future and toward the past, respectively. The positive imaginary solution of the eingenvalue problem,

$$\langle x^0 x^3|H_{\text{osc}}|p_0 n\rangle = ieE\left(n + \frac{1}{2}\right)\langle x^0 x^3|p_0 n\rangle \tag{24}$$

corresponds to the generalized eingenstate

$$\langle x^0 x^3|p_0 n\rangle = e^{ip_0 x^0}\varphi_n(x^3 + p_0/eE) \tag{25}$$

representing a decaying Gamow vector

$$|p_0 n(s)\rangle = e^{-eE(n+1/2)s}|p_0 n\rangle \tag{26}$$

for $s > 0$.[6]

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[6]The positive frequency choice corresponds to positive poles of $R(z)$ in the second sheet of the upper half-plane. This analytical extension of the resolvent operator corresponds to the retarded (causal) evolution propagator in proper time. Then decaying Gamow vectors are only defined for positive times.

Using mass 'eingenstates' $|p_1 p_2, p_0 n\rangle$ of H, we can easily evaluate the matrix element (see Appendix)

$$\langle x|e^{iHs}|x\rangle = -\frac{i}{(4\pi s)^2}\left[\frac{eEs}{\sinh(eEs)}\right]$$

Expanding it in power series of the coupling constant

$$\langle x|e^{iHs}|x\rangle = -\frac{i}{(4\pi s)^2}\left[1 - \frac{(eEs)^2}{2} + \frac{7}{360}(eEs)^4 + \cdots\right]$$

and replacing the third term in the integral, we finally have

$$L_{\text{eff}}(x) = \cdots - \frac{7}{360}\frac{e^4 E^4}{(4\pi)^2}\int_0^\infty s e^{-i(m^2 - i\epsilon)s}ds = \cdots + \frac{7}{360}\frac{e^4 E^4}{m^4}$$

which coincides with the expression obtained by Schwinger (1951) for the Heisenberg–Euler correction in the spinless case.

Now let us discuss the particle creation process associated with the imaginary part of

$$L_{\text{eff}}(x) = -\frac{1}{(4\pi)^2}\int_0^\infty \frac{eEs}{s^3 \sinh(eEs)} e^{-i(m^2 - i\epsilon)s} \, ds + C(x)$$

We can analytically continue the integrand to the lower half-plane; then the integral along the positive real axis becomes

$$L_{\text{eff}}(z) = -\frac{1}{(4\pi)^2}\int_{\Gamma_0} \frac{eE}{z^2 \sinh(eEz)} e^{-im^2 z} \, dz + C(x)$$

where Γ_0 is a path with the same endpoints. The integrand has poles in $z_{\pm n} = \pm in\pi/eE$.[7] Using the residues theorem, we can rewrite $L_{\text{eff}}(z)$ as

$$L_{\text{eff}}(z) = -\frac{1}{(4\pi)^2}\int_{\Gamma_{-1}} \frac{eE}{z^2 \sinh(eEz)} e^{-im^2 z} \, dz + C(x) - 2\pi i \operatorname{Res}(z_{-1})$$

where the path Γ_{-1} has the same endpoints as Γ_0 in such a way that the closed counterclockwise contour $\Gamma_{-1} \cup (-\Gamma_0)$ encircles the first pole in the

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[7] Note also that the resolvent

$$R(z) = \frac{1}{\pi^2 - z}$$

has a cut on the positive real axis and poles $z_n = \pm ieE(n + 1/2)$ in the positive imaginary one. Since the effective Lagrangian can be written as $L_{\text{eff}}(z) = -i\langle x| \ln R(z)|x\rangle + C(x)$, the poles of the resolvent are related to the poles of the integrand.

negative imaginary axis, $z_{-1} = -i\pi/eE$. The probability of creating one pair[8] per space-time volume is given by

$$w(x) = \text{Im } L_{\text{eff}}(x) = -2\pi \text{ Res}(z_{-1})$$

Thus we finally have

$$w(x) = \frac{(eE)^2}{8\pi^3} \exp\left(-\frac{\pi m^2}{eE}\right) \tag{27}$$

according to Sauter's estimation.

5. GAMOW VECTORS ASSOCIATED WITH PARTICLE CREATION BY AN EXTERNAL FIELD

In Section 4 we saw that an inverted harmonic oscillator structure appears in the Hamiltonian. We claim that the instability of the vacuum in the field-theoretic language has its correlate in this simple unstable system at the level of the off-shell theory. We noticed that this inverted oscillator has a pure imaginary frequency $\omega = \pm ieE$ (antioscillator). This leads to a 'complex eigenvalue' problem, which requires some technical points that we discuss in this section.

We can solve the eingenvalue problem for this system from the solutions of the harmonic oscillator, obtaining two sets of 'complex eigenvalues' (which do not take part of the spectrum of H_{osc})

$$\langle x^0 x^3 | H_{\text{osc}} | p_0 n \rangle = ieE\left(n + \frac{1}{2}\right)\langle x^0 x^3 | p_0 n \rangle \tag{28}$$

$$\langle x^0 x^3 | H_{\text{osc}} | \widetilde{p_0 n} \rangle = -ieE\left(n + \frac{1}{2}\right)\langle x^0 x^3 | \widetilde{p_0 n} \rangle \tag{29}$$

where

$$\langle x^0 x^3 | p_0 n \rangle = e^{ip_0 x^0} \varphi_n(x^3 + p_0/eE) \tag{30}$$

$$\langle x^0 x^3 | \widetilde{p_0 n} \rangle = e^{ip_0 x^0} \widetilde{\varphi_n}(x^3 + p_0/eE) \tag{31}$$

where φ_n and $\tilde{\varphi}_n$ contain Hermite polynomials of complex argument. The 'eigenvalues' correspond to complex poles $z_n = \pm ieE(n + 1/2)$ along the

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[8]The contribution corresponding to creating two pairs can be obtained taking a path which encircles the pole z_{-2}, and so on.

positive/negative imaginary axis of the resolvent operator $R(z)$. It was demonstrated (Castagnino *et al.*, 1997) that the corresponding 'eigenvectors' form a biorthogonal set, i.e.,

$$\langle \widetilde{p_0 n} | p'_0 m \rangle = \delta(p_0 - p'_0)\delta_{nm} \quad \text{in} \quad \Phi_+ \tag{32}$$

$$\langle p_0 n | \widetilde{p'_0 m} \rangle = \delta(p_0 - p'_0)\delta_{nm} \quad \text{in} \quad \Phi_- \tag{33}$$

$$\int dp_0 \sum_{n=0}^{\infty} |p_0 n\rangle\langle \widetilde{p_0 n}| = I \quad \text{in} \quad \Phi_+ \tag{34}$$

$$\int dp_0 \sum_{n=0}^{\infty} |\widetilde{p_0 n}\rangle\langle p_0 n| = I \quad \text{in} \quad \Phi_- \tag{35}$$

The eigenvectors (30) and (31) correspond to generalized eigenvectors of H_{osc} in adequate rigged Hilbert spaces:

$$\langle H_{osc} \phi | p_0 n \rangle = \langle \phi | H_{osc} | p_0 n \rangle = ieE\left(n + \frac{1}{2}\right)\langle \phi | p_0 n \rangle \tag{36}$$

$$\langle H_{osc} \psi | \widetilde{p_0 n} \rangle = \langle \psi | H_{osc} | \widetilde{p_0 n} \rangle = -ieE\left(n + \frac{1}{2}\right)\langle \psi | \widetilde{p_0 n} \rangle \tag{37}$$

where $\phi \in \Phi_+$ and $\psi \in \Phi_-$, since $H_{osc}^{+} = H_{osc}$ is continuous on $\Phi_{\pm}$. The test spaces Φ_+ and Φ_- are defined by (Castagnino *et al.*, 1997)

$$\Phi_+ = \{\phi \in \mathscr{S}/\langle v|\phi\rangle \in \mathfrak{L}\} = \{\phi \in \mathscr{S}/\langle u|\phi\rangle \in \mathfrak{K}\} \tag{38}$$

$$\Phi_- = \{\psi \in \mathscr{S}/\langle v/\psi\rangle \in \mathfrak{K}\} = \{\psi \in \mathscr{S}/\langle u|\psi\rangle \in \mathfrak{L}\} \tag{39}$$

where $\mathfrak{K}$ is the subset of Schwarz functions ($\mathscr{S}$) of compact support and $\mathfrak{L}$ is the subset of Schwarz integer functions of exponential order, restricted to the real axis. $\{|v\rangle\}$ and $\{|u\rangle\}$ are two representations constructed with the generalized eigenvectors of the operators

$$v = \frac{1}{\sqrt{2}}\left[\frac{p_3}{\sqrt{eE}} + \sqrt{eE}\left(x_3 + \frac{p_0}{eE}\right)\right] \tag{40}$$

$$u = \frac{1}{\sqrt{2}}\left[\frac{p_3}{\sqrt{eE}} - \sqrt{eE}\left(x_3 + \frac{p_0}{eE}\right)\right] \tag{41}$$

which are the analogues of the creation and annihilation operators for the harmonic oscillator. Therefore we can construct a pair of rigged Hilbert spaces:

$$\Phi_\pm \subset \mathcal{L}^2(\mathbb{R}) \subset \Phi_\pm^\times \qquad (42)$$

where $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space, $\Phi_\pm$ are dense subsets of $\mathcal{L}^2(\mathbb{R})$ with their own complete nuclear topology, and $\Phi_\pm^\times$ are the dual spaces of $\Phi_\pm$. The evolution operator $U = e^{iHs}$ is continuous on $\Phi_\pm$ and such that $U^\dagger\Phi_\pm \subset \Phi_\pm$, for $s \gtrless 0$ only. Thus we can obtain the evolution in s of the pair of Gamow vectors as

$$|p_0 n(s)\rangle = e^{-eE(n+1/2)s}|p_0 n\rangle \qquad \text{for} \quad s > 0 \qquad (43)$$

$$\overline{|p_0 n(s)\rangle} = e^{eE(n+1/2)s}\overline{|p_0 n\rangle} \qquad \text{for} \quad s < 0 \qquad (44)$$

which are functionals in $\Phi_\pm^\times$, respectively. We see that the pair of Gamow vectors represent a decaying state toward the future and a growing state from the past, respectively. These Gamow vectors are related via the s-time reversal Wigner operator K,[9] since it can be proved (Castagnino *et al.*, 1997) that K: $\Phi_\pm \to \Phi_\mp$, and therefore

$$K: \quad \Phi_\pm^\times \to \Phi_\mp^\times \qquad (45)$$

We have seen that the unitary temporal evolution splits into two semigroups. Let us interpret the physical meaning of such a splitting. Coming back to the off-shell propagator, let us evaluate it for the Hamiltonian (22):

$$G+(x, y, s)$$

$$= \theta(s)\langle x|e^{iHs}|y\rangle$$

$$= \theta(s) \int dp_1 \int dp_2 \int dp_0 \sum_{n=0}^\infty \langle x|e^{iHs}|p_1 p_2 p_0 n\rangle \overline{\langle p_1 p_2 p_0 n|y\rangle} \qquad (46)$$

where we have used the completeness relation (34), since their generalized eigenvectors are well defined only for $s > 0$. It shows that the retarded condition in time s (Feynman prescription) is satisfied only for Gamow vectors decaying toward the future. Similarly, the Dyson prescription is related to growing Gamow vectors.

We can conclude that the choice of Feynman or Dyson prescriptions is *a priori* a conventional matter when we consider the whole universe (or a

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[9]The Wigner operator K is such that conjugates the wave function of the Klein–Gordon equation and coincides with the charge conjugation operator for this equation. The s-reversal operation of (18) is given by the operator S such that $S\Psi(x, s) = K\Psi(x, -s)$, which coincides with the generalization of the charge conjugation operation of (18).

closed isolated system). That is, for example, the Feynman boundary condition for which particles propagate toward the future and antiparticles toward the past in coordinate time is correlated with the Gamow vectors decaying toward the future in proper time. Then we have an unavoidable proper time asymmetry, coming from a proper time symmetrical theory which splits the dynamical evolution into two semigroups. But once we have made the Feynman choice, 'proper time' asymmetry is a substantial thing providing a privileged direction of time, the one in which Gamow vectors decay. The Dyson choice only leads to a specular world, in which we can be living just now, if we conventionally interchange the role of past and future.

APPENDIX

The matrix element $\langle x | e^{iHs} | x \rangle$ can be factorized as the matrix element for two free particles and an upside-down harmonic oscillator

$$\langle x^1 | e^{-i(p_1)^2 s} | x^1 \rangle \langle x^2 | e^{-i(p_2)^2 s} | x^2 \rangle \langle x^0 x^3 | e^{i(H_{\text{osc}})s} | x^0 x^3 \rangle$$

The first factor gives a contribution

$$\langle x^1 | e^{-i(p_1)^2 s} | x^1 \rangle = \int_{-\infty}^{\infty} \langle x^1 | p^1 \rangle \langle p_1 e^{-i(p_1)^2 s} | x^1 \rangle \, dp_1$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i(p_1)^2 s} \, dp_1 = \frac{1}{2\pi} \sqrt{\frac{\pi}{is}}$$

with the same factor for the second one. The third factor can be computed in a similar way,

$$\langle x^0 x^3 | e^{i(H_{\text{osc}})s} | x^0 x^3 \rangle$$

$$= \sum_n \int_{-\infty}^{\infty} \langle x^0 x^3 | e^{i(H_{\text{osc}})s} | p_0 n \rangle \langle \widetilde{p_0 n} | x^0 x^3 \rangle \, dp_0$$

$$= \sum_n \int_{-\infty}^{\infty} e^{-eE(n+1/2)s} \varphi_n(x^3 + p_0/eE) \widetilde{\varphi}_n(x^3 + p_0/eE) \, dp_0$$

$$= \frac{eE}{2\pi} \sum_n e^{-eE(n+1/2)s} \int_{-\infty}^{\infty} \widetilde{\varphi}_n(u) \varphi_n(u) \, du$$

$$= \frac{eE}{2\pi} \left[\frac{1}{2 \sinh(eEs)} \right]$$

Collecting our partial results, we finally have

$$\langle x | e^{iHs} | x \rangle = -\frac{i}{(4\pi s)^2} \left[\frac{eEs}{\sinh(eEs)} \right]$$

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